

Area under a polar curve by the Monte Carlo Simulations

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Abstract— The Monte Carlo method is a computerized mathematical algorithm that generate pseudorandom numbers following a particular probability distribution (viz lognormal, normal, uniform, Bernoulli, Rayleigh, Triangular and so on). Monte Carlo simulations find applications in a variety of fields including geological statistics, radiation materials sciences, business models, inventory management and so on.

Monte Carlo Methods are largely used in optimization, incorporating real-world uncertainty in financial models in order to make more accurate decisions, and numerical integration. This paper shall be focusing on using Monte Carlo methods in numerical integration. Numerical integration encompasses evaluation of definite integrals which are geometrically analogous to the area under a curve (polar, quadratic, linear, exponential, logarithmic and so on). Polar curves are closed figures, and it is possible for it to be inscribed in a rectangle with sides created by the tangents drawn perpendicular to the coordinate axes. A Monte Carlo simulation can then be run to generate pseudorandom numbers based on a given domain. These shall represent the coordinates of various points within or on the rectangle created. The ratio of the points outside and within the polar curve shall be used to calculate its area.

Index Terms— Approximation, Integrals, Monte Carlo simulations, Polar coordinates, Polar curves, Pseudorandom number generators, tangent lines



1 INTRODUCTION

In order to determine the value of pi using Monte Carlo Simulation, quadrant is inscribed inside a unit square. Then, two sets of pseudorandom numbers are generated, one representing the x-coordinate and the other representing the y-coordinate. Taken together, they shall represent a specific point on the x-y Cartesian Plane subject to the constraint:

$$\begin{aligned} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{aligned}$$

Thus, if $x^2 + y^2 < 1$, the point is considered to be inside the inscribed quadrant. These points have been denoted in red. If $x^2 + y^2 > 1$, the point lies outside the inscribed quadrant and is denoted in blue.

After running nearly 10,000 simulations, while simultaneously keeping track of the number of points falling inside the quadrant. The ratio of the points inside and the total number of simulations is nearly equal to

$$\frac{\pi(1)^2}{\frac{4}{1^2}} = \frac{\pi}{4}$$

Multiplying this by 4 gives the value of π

2 CONSTRAINTS

While calculating the values of constants or areas using the Monte Carlo Simulations, the following constraints apply:

- 1) The pseudorandom number generator must follow a uniform distribution
- 2) A large number of simulations must be run in order to gain a better approximation

3 GENERALIZING THE DISCUSSION TO CLOSED

AREAS

The method used to approximate the value of pi shall be generalized to calculating the area enclosed by the polar curves. Indirectly, this shall equal the definite integral used to evaluate the area enclosed by them. Hence, this method is at par with similar methods of approximation such as the Left, Right, Trapezoidal, and Mid-Point Riemann Sums. sections upper case Arabic numerals, then upper case Arabic numerals, separated by periods. Initial paragraphs after the section title are not indented. Only the initial, introductory paragraph has a drop cap.

4 DETERMINING THE AREA OF A POLAR CURVE USING THE MONTE CARLO METHODS

Consider the equation of the simple polar curve below:

$$r = 2 + 2\cos\theta$$

Sketching the graph of this polar curve on the standard x-y cartesian plane, we obtain a cardioid as:

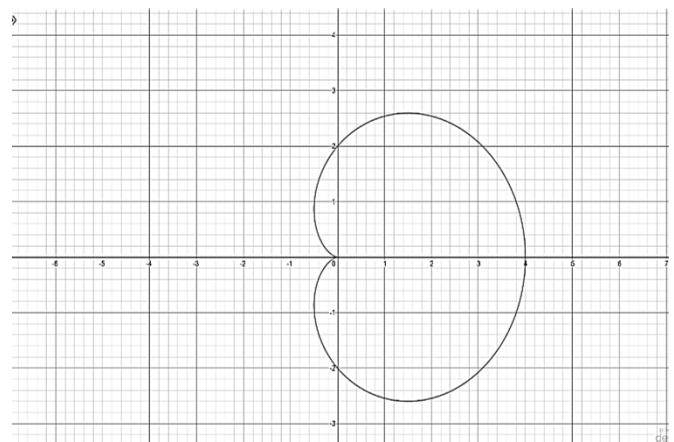


Fig. 1. The graph of the polar curve $r = 2 + 2\cos\theta$ on the Cartesian plane

The area of this cardioid is given by the definite integral:

$$S = \int_0^\pi (2 + 2\cos\theta)^2 d\theta$$

The integral can be evaluated using pseudo-random number generation in a 10000-time Monte Carlo Simulation.

The first step is determining the boundaries of the cardioid: This can be achieved by finding those points on the cardioid where the tangent line is either normal or parallel to the x-axis. Accordingly,

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ \frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = m \text{ (slope of tangent line)} \end{aligned}$$

Using the following equation:

$m=0$ and $m=\text{undefined}$ (resolved by equating the denominator to 0) we get the values of θ for which we have horizontal and vertical tangents. (limiting values).

Substituting these values in the equation of the polar curve we get the appropriate value of r (limiting values) which is then plugged in to get the values of x and y .

Thus, we can obtain the horizontal and vertical tangent line equations manually.

The intersection of these lines forms a rectangle that perfectly inscribes the cardioid. In the graph shown below the horizontal and vertical tangent lines have been marked.

Horizontal tangents: $y = \pm 2.6$

Vertical tangents: $x = 4, x = -0.5$

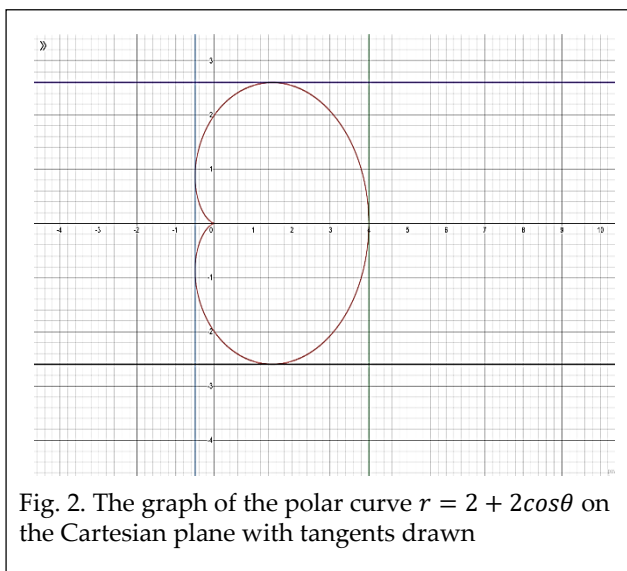


Fig. 2. The graph of the polar curve $r = 2 + 2\cos\theta$ on the Cartesian plane with tangents drawn

coordinates.

Each pseudo-random point is then converted to polar coordinates as:

Let a random point (x,y) be generated subject to the constraint:

$$\begin{aligned} x_\delta &< x < x_\Delta \\ y_\delta &< y < y_\Delta \end{aligned}$$

Where x_δ and x_Δ represent the vertical tangent lines (side limits) and y_δ and y_Δ represent the horizontal tangent lines (top-bottom limits).

$$(x, y) \equiv (r, \theta) \equiv (\sqrt{x^2 + y^2}, \tan^{-1} \frac{y}{x})$$

Substituting the value of θ in the polar equation shall help determine whether the point lies inside or outside the polar curve:

The area of the polar curve shall be thus given by:

$$S = \int_0^\pi (2 + 2\cos\theta)^2 d\theta \approx \frac{\text{no. of points inside the polar curve}}{10000} (x_\Delta - x_\delta)(y_\Delta - y_\delta)$$

5 CONCLUSION

The primary conclusion of drawn from the research paper is applicable to geometrical figures beyond polar curves too. This can be a useful method to approximate the areas of extremely irregular 2-dimensional or 3-dimensional figures in the Cartesian Plane.

Now, a Monte Carlo Simulation is run for 10000 cartesian